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Propositional Systems and Measurements—I

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Abstract

A direct approach to propositional systems, influenced by the exposition of Jauch & Piron (1968), is formulated. Further motivations and interpretations in experimental terms are given and worked out explicitly. The orthocomplementation is introduced independent of weak modularity by two postulates on the set of questions. Weak modularity and atomicity are reduced to requirements on the set of preparation procedures. It is shown that preparative measurements are not needed in order to impose the covering law. The latter can also be reduced to a postulate on the set of preparations.

I

Birkhoff & Neumann (1936) have shown how a logic of propositions on quantum systems can be used to derive the usual structure of quantum mechanics. They assumed this logic to be a modular lattice. As pointed out by Piron (1964) and Jauch (1968) this assumption seems to be unrealistic. Firstly, in the usual interpretation of quantum mechanics propositions correspond to the closed subspaces of a Hilbert space which do not form a modular lattice. Secondly, it seems questionable whether a physical system can be localisable if the logic of propositions is a modular lattice in which the distributive law does not hold. A new axiomatic scheme which does not require modularity has been given by Piron (1964). It is reasonable because the axioms imply the logic to be isomorphic to some sublattice of closed subspaces of a Hilbert space.

In formulating their axioms Birkhoff & Neumann (1936) as well as Piron (1964) have looked for some weaker structure than phase space logic of classical mechanics preferring, on the other hand, the strongest one into which the usual quantum mechanics can be fitted. This fruitful method has given appropriate axioms and an interpretation resulting from comparison of the new structure with that of classical mechanics. As phase space in classical mechanics is itself a very abstract construct, the concepts used in that interpretation are not very suitable in deciding which axioms seem obvious from the outset and which seem to make serious statements on reality.

A step forward in interpreting the axioms in less abstract terms describing the experimental situation is made by Jauch & Piron (1969), Jauch (1968) and Piron (1972). The concept of a question as any measurement prescription for experiments with only two possible outcomes is introduced as basic. Propositions are now understood as classes of equivalent questions. The axioms arise as requirements on the set of questions, state preparations, and operations on states of a physical system which actually can be performed.

This axiomatic scheme can be divided into two parts. The first part implies the lattice of propositions, now called propositional system, to be a CROC, i.e. a complete, orthocomplemented, and weakly modular lattice, and makes explicit assumptions neither on state preparations nor on operations on states. The second part makes assumptions on state preparations in order to imply the lattice atomic, and, finally, assumptions on operations in order to imply the covering law.

Some conceptual difficulties are inherent in the expositions of Jauch and Piron. Motivations and interpretations given are partly very short or only sketched and have led to misunderstandings (Ochs, 1972a, 1972b). Another point is the introduction of the compatible complement (Piron, 1972). It is introduced as some kind of commensurable or, more precisely, as a coexistent complement. One expects some motivation to axiom *P* from this, but no connection is given. Take axiom *C* (Piron, 1972, p. 294) to state: 'For each proposition there is a choice of a complement which we call the compatible choice'. Then the Theorem (Piron, 1972, p. 295) 'Let $\mathscr{L}$ be a CROC. If one interprets the orthocomplement as a compatible complement, then $\mathscr{L}$ satisfies axioms *C* and *P*' would serve equally well as a motivation for axioms *C* and *P*. The requirement $a = [\alpha]$ and $a' = [\alpha^{\sim}]$ is not used in the proof. That the compatible complement is introduced in this way is neither motivated nor used in the sequel by Piron. Because there may be several coexistent complements unless axiom *P* is stated, this does not even fix the choice of a unique complement.

In this paper an approach to propositional systems is given, which works out explicitly the connection of the axioms of Piron with postulates on real preparations and measurements. We begin with a given set of preparation procedures and a given set of questions and derive the structure of the propositional system from postulates on the extension of these sets.

We shortly review, along the lines of Piron's reasoning, how a natural structure of the set of questions gives the propositional system the structure of a complete lattice (Section I). Then we try to interpret the propositions as statements about properties of physical systems. This motivates two postulates on the extension of the set of questions, which fixes an orthocomplementation independent of weak modularity (Section III). Weak modularity and atomicity are reduced to two postulates on the extension of the set of preparation procedures (Sections IV and V).

We are then left with the problem of motivating the covering law, which is sufficient to carry through the coordinatisation procedure connecting the usual quantum mechanics with the propositional calculus. We think the assump tion of ideal questions of the first kind is not a realistic basis for motivating the covering law and replace it by a superposition postulate which we show to be an equivalent basis for this purpose (Section VI).

At the end of Section VI we mention a subsequent paper containing results on the problem of how to formulate measuring processes if the covering law does not hold for the object system.

Another motive for writing this paper was the fact that the great mathematical simplicity of the direct approach to propositional systems makes it accessible also to non-specialists in foundations of quantum mechanics. This is in contrast to the probabilistic approach, which presupposes the study of duality theory of partially ordered linear spaces.

II

Let us be given a physical system, some set $\tilde{\mathscr{S}}$ of preparation procedures for it, and some set $\tilde{\mathscr{Q}}$ of questions, such that any $\alpha \in \tilde{\mathscr{Q}}$ can be combined with any $s \in \tilde{\mathscr{S}}$ to give a prescription for experiments with only two outcomes, 'yes' and 'no', possible. We require that the experiments can be carried through arbitrarily often and call a question 'true' for a preparation procedure if the probability for outcome yes is one. Truth is only stated for preparations in which any single system prepared by the prescribed procedure determines the outcome yes.† So the truth of a question is a statement on single systems in contrast to the statement that a probability is unequal to zero or one, which applies only to ensembles.

We define a completion $\mathscr{Q}$ of $\tilde{\mathscr{Q}}$ by the following steps:

(i) Adjoin to $\tilde{\mathscr{Q}}$ a question I which on any $s \in \tilde{\mathscr{S}}$ gives the outcome yes, and denote the join by $\bar{\mathscr{Q}}$.

(ii) For $\alpha \in \bar{\mathscr{Q}}$ define $\nu\alpha$ by: The outcome of $\nu\alpha$ is yes iff the outcome of α is no, and the outcome of $\nu\alpha$ is no iff the outcome of α is yes. Adjoin to $\bar{\mathscr{Q}}$ all $\nu\alpha$ with $\alpha \in \bar{\mathscr{Q}}$, and denote the join by $\hat{\mathscr{Q}}$.
We denote νI by Φ.

(iii) Let $\{\alpha_i\}_{i \in J}$ denote an arbitrary subset of $\hat{\mathscr{Q}}$. Define the question $\Pi_{i \in J}\, \alpha_i$ by : Carry through one *randomly* chosen α_i and take the result to be that of $\Pi_{i \in J}\, \alpha_i$. Adjoin $\Pi_{i \in J}\, \alpha_i$ for all subsets of $\hat{\mathscr{Q}}$ to $\hat{\mathscr{Q}}$, and denote the join by $\mathscr{Q}$.
We denote $\Pi_{i \in \{1,2\}}\, \alpha_i$ by $\alpha_1\alpha_2$.

One easily checks that $\nu\, \Pi_{i \in J}\, \alpha_i$ and $\Pi_{i \in J}\, \nu\alpha_i$ prescribe exactly the same manipulations for arriving at a yes or no result. Thus the outcome of any single experiment does not depend on whether we interpret it as $\Pi_{i \in J}\, \nu\alpha_i$ or as $\nu\, \Pi_{i \in J}\, \alpha_i$. So we shall identify both.‡ Then $\alpha \in \mathscr{Q}$ implies $\nu\alpha \in \mathscr{Q}$. ν is a bijection on $\mathscr{Q}$.

† More precisely, this can only be stated in the sense of stochastic convergence: If α is true, the outcome no can arise at most for a finite subset of a sequence of infinitely many experiments.

‡ $(\alpha\,.\,\nu\alpha) = \nu(\alpha\,.\,\nu\alpha)$ is no contradiction because $\alpha\,.\,\nu\alpha$ is unequal to Φ.

The subset of preparation procedures for which a given α is true will be denoted by $\mathcal{O}(\alpha)$, $\mathcal{O}(\alpha) \subseteq \tilde{\mathcal{S}}$. Propositions are then defined by

$$[\alpha] =: \{\gamma \mid \mathcal{O}(\gamma) = \mathcal{O}(\alpha)\}.$$

We call $[\alpha]$ 'true' for a preparation procedure if there is a $\gamma \in [\alpha]$ which is true and denote by $\mathcal{O}([\alpha])$ the subset of preparation procedures for which a given $[\alpha]$ is true. Then by $[\alpha] \mapsto \mathcal{O}([\alpha])$ the set of propositions, which we denote by $[\mathcal{Q}]$, is mapped injectively into the power set of $\tilde{\mathcal{S}}$. The statement '$[\beta]$ is true whenever $[\alpha]$ is true' with respect to $\tilde{\mathcal{S}}$ is expressed equivalently by $\mathcal{O}([\alpha]) \subseteq \mathcal{O}([\beta])$. We write $[\alpha] \leqslant [\beta]$ iff

$$\mathcal{O}([\alpha]) \subseteq \mathcal{O}([\beta]).$$

This introduces a partial ordering on $[\mathcal{Q}]$ which means '$[\alpha]$ implies $[\beta]$'.

From (iii) in the definition of $\mathcal{Q}$ it follows that

$$\mathcal{O}([\prod_{i \in J} \alpha_i]) = \bigcap_{i \in J} \mathcal{O}([\alpha_i]) = \text{g.l.b.}\{\mathcal{O}([\alpha_i])\}_{i \in J}.$$

From this one easily concludes that the greatest lower bound exists for any subset of $[\mathcal{Q}]$. As $[\mathcal{Q}]$ itself has an upper bound, $[I]$, a well-known theorem states that the lowest upper bound exists for any subset of $[\mathcal{Q}]$. Hence $([\mathcal{Q}], \leqslant)$ is a complete lattice.

We denote the lattice operations meet and join in $([\mathcal{Q}], \leqslant)$ by $\wedge$ and $\vee$ respectively. $[\alpha] \wedge [\beta]$ can be interpreted by '$[\alpha]$ and $[\beta]$' and $\wedge$ is a conjunction in the usual sense because of

$$\mathcal{O}([\alpha] \wedge [\beta]) = \mathcal{O}([\alpha]) \cap \mathcal{O}([\beta]).$$

In contrast $[\alpha] \vee [\beta]$, which we interpret as '$[\alpha]$ or $[\beta]$', is not a disjunction in the usual sense. In fact, we only have $\mathcal{O}([\alpha]) \cup \mathcal{O}([\beta]) \subseteq \mathcal{O}([\alpha] \vee [\beta])$, so $[\alpha] \vee [\beta]$ may be true although neither $[\alpha]$ nor $[\beta]$ is true.

This construction of $([\mathcal{Q}], \leqslant)$ as a complete lattice needs no assumption about the extension of the given sets $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{Q}}$. Such assumptions will be made in order to impose further structures.

III

A proposition $[\alpha]$ is true for any single system which is prepared by a procedure in $\mathcal{O}([\alpha])$. This presumes to interpret $[\alpha]$ in a predicate calculus, stating that the property $([\alpha])$ is present iff $[\alpha]$ is true. The property $([\alpha])$ is defined by the class of equivalent measurement procedures $[\alpha]$.

By this we are given a set of procedures to prepare systems which have property $([\alpha])$ and a set of questions, namely $[\alpha]$, by which this property may be detected. But we do not know what the absence of $([\alpha])$ means. If $[\alpha]$ is not true for a preparation procedure, two cases may arise. Either there is a γ,

$\gamma \in \mathcal{Q}$, $[\alpha] \leqslant [\gamma]$ such that $[\nu\gamma]$ is true, or for all such γ $[\nu\gamma]$ is not true. We define $([\alpha])$ to be absent for any physical system prepared by a procedure for which the first case arises.

In the second case we do not know in advance whether, for a single system, the outcome will be yes for γ or $\nu\gamma$. Both are possible for any γ, $[\alpha] \leqslant [\gamma]$. This does not imply that a property may be neither present nor absent for a given physical system. Such a conclusion would presuppose a basis for assuming any system prepared by a certain procedure to have the 'same properties'. A basis for this would be given by a suitable concept of pure states.

Until now there need not be a proposition in $[\mathcal{Q}]$ which defines absence of $([\alpha])$ because there may be no δ in

$$\mathcal{F}([\alpha]) =: \{\gamma \mid \gamma \in \mathcal{Q}, [\alpha] \leqslant [\gamma]\}$$

such that $[\nu\delta]$ is maximal. In other words, absence of a property is not, in general, a property in the sense introduced above unless we require

Postulate 1. For any $\alpha, \alpha \in \mathcal{Q}$, there is a $\delta, \delta \in \mathcal{F}([\alpha])$, such that $\gamma \in \mathcal{F}([\alpha])$ implies $[\nu\gamma] \leqslant [\nu\delta]$.

One easily checks $[\nu\delta]$ to be unique. So we may define the mapping

$$\varphi: \quad [\mathcal{Q}] \to [\mathcal{Q}]$$
$$[\alpha] \mapsto [\nu\delta].$$

Some properties which are proved in the appendix are listed:

(i) $[\alpha] \leqslant \varphi^2[\alpha] \qquad (\varphi^2 =: \varphi \circ \varphi)$.
(ii) For any $\alpha, \alpha \in \mathcal{Q}$, there is a $\epsilon, \epsilon \in \mathcal{Q}$, such that $\epsilon \in \varphi^2[\alpha]$, and $\nu\epsilon \in \varphi[\alpha]$.
(iii) $\varphi^3[\alpha] = \varphi[\alpha]$.
(iv) $\varphi[\alpha] \leqslant \varphi[\beta]$ is equivalent to $\varphi^2[\alpha] \geqslant \varphi^2[\beta]$.
(v) $\varphi[\alpha] \wedge \varphi^2[\alpha] = [\Phi]$.
(vi) $\varphi^2(\varphi[\alpha] \vee \varphi^2[\alpha]) = [I]$.

By Postulate 1 we have, to any property $([\alpha])$, a counterpart, $(\varphi[\alpha])$, which means '$([\alpha])$ is absent'. But (i) leaves open whether the absence of $(\varphi[\alpha])$ implies $[\alpha]$. Moreover, equality in (i) would imply φ to be an orthocomplementation which is, by (ii), compatible in the sense of Piron.

Theorem 1. φ is an orthocomplementation on $([\mathcal{Q}], \leqslant)$ iff $\varphi([\mathcal{Q}]) = [\mathcal{Q}]$.

Proof. Let $\varphi([\mathcal{Q}]) = [\mathcal{Q}]$, then $[\alpha] \in [\mathcal{Q}]$ implies $[\alpha] = \varphi[\beta], \beta \in \mathcal{Q}$. So $\varphi^2[\alpha] = [\alpha]$.

By (iii) through (vi) φ is an orthocomplementation. The converse is trivial.

Postulate 2. $\varphi([\mathcal{Q}]) = [\mathcal{Q}]$.

So an orthocomplementation is imposed on the propositional system by two requirements on the extension of the set $\mathcal{Q}$ of questions. Further structures will now be imposed by requirements on the extension of the set $\mathcal{S}$ of preparation procedures.

IV

There are well-known criteria for an orthocomplemented lattice to be weakly modular (Rose, 1964). We show one of them in the following form.

Theorem 2. $([\mathcal{Q}], \leqslant, \varphi)$ is weakly modular if and only if $[\alpha] \leqslant [\beta]$ implies that $\varphi[\alpha] \wedge [\beta] = [\Phi]$ is equivalent to $[\alpha] = [\beta]$.

Proof. Let $([\mathcal{Q}], \leqslant, \varphi)$ be weakly modular, $[\alpha] \leqslant [\beta]$, and $\varphi[\alpha] \wedge [\beta] = [\Phi]$. Hence $[\beta] = [\alpha] \vee (\varphi[\alpha] \wedge [\beta]) = [\alpha]$. Conversely, let the condition hold, and $[\gamma] \leqslant [\beta]$. Then $[\alpha] =: ([\beta] \wedge \varphi[\gamma]) \vee [\gamma] \leqslant [\beta]$. Moreover, $\varphi[\alpha] \wedge [\beta] = \varphi([\beta] \wedge \varphi[\gamma]) \wedge \varphi[\gamma] \wedge [\beta] = [\Phi]$. So $[\alpha] = ([\beta] \wedge \varphi[\gamma]) \vee [\gamma] = [\beta]$.

The condition of the theorem can easily be interpreted. If $[\alpha] \leqslant [\beta]$, then $[\alpha] \neq [\beta]$ is equivalent to $\mathcal{O}(\varphi[\alpha]) \cap \mathcal{O}([\beta]) \neq \phi$, i.e. there is at least one preparation procedure such that property $([\alpha])$ is absent and property $([\beta])$ is present for any physical system prepared in the prescribed manner. In order to impose weak modularity on $([\mathcal{Q}], \leqslant, \varphi)$ we have to require:

Postulate 3. If $[\alpha] \leqslant [\beta]$, then $[\alpha] \neq [\beta]$ is equivalent to $\mathcal{O}(\varphi[\alpha]) \cap \mathcal{O}([\beta]) \neq \phi$.

V

We shall now consider sets of preparation procedures and introduce a concept of pure states. For a subset $\mathfrak{s}$ of $\tilde{\mathcal{S}}$ we define

$$t(\mathfrak{s}) =: \{[\alpha] \mid [\alpha] \text{ is true if } s \in \mathfrak{s}\}.$$

If $[\alpha] \in t(\mathfrak{s})$, then $\varphi[\alpha] \notin t(\mathfrak{s})$. This follows from $\mathcal{O}([\alpha]) \cap \mathcal{O}(\varphi[\alpha]) = \phi$. From $[I] \in t(\mathfrak{s})$ we have $[\Phi] \notin t(\mathfrak{s})$. Moreover $[\alpha] \in t(\mathfrak{s})$ and $[\beta] \in t(\mathfrak{s})$ implies

$$[\alpha] \wedge [\beta] \in t(\mathfrak{s}).$$

If we denote the power set of $\tilde{\mathcal{S}}$ by $\mathcal{P}(\tilde{\mathcal{S}})$, the mapping

$$\tilde{a}: \quad t(\mathcal{P}(\tilde{\mathcal{S}})) \to [\mathcal{Q}] \backslash \{[\Phi]\}$$
$$t(\mathfrak{s}) \mapsto \bigwedge_{t(\mathfrak{s})} [\alpha]$$

is bijective. It is surjective because $\tilde{a} \circ t(\mathcal{O}([\gamma])) = [\gamma]$, $[\gamma] \in [\mathcal{Q}] \backslash \{[\Phi]\}$. It is injective because $\tilde{a} \circ t(\mathfrak{s}) = \tilde{a} \circ t(\mathfrak{u})$ implies $t(\mathfrak{s}) = t(\mathfrak{u})$, which is easily checked by $t(\mathfrak{s}) = \{[\alpha] \mid \tilde{a} \circ t(\mathfrak{s}) \leqslant [\alpha]\}$.

A set $\mathfrak{s}$ of preparation procedures is given, if certain macroscopic conditions are only approximately fulfilled in the experimental set-up. Any $s \in \mathfrak{s}$ which may arise in a single experiment works equally well if one needs only systems with property $(\tilde{a} \circ t(\mathfrak{s}))$.

We introduce classes of equivalent sets of preparation procedures by

$$[\mathfrak{s}] =: \{\mathfrak{u} \mid \mathfrak{u} \in \mathcal{P}(\tilde{\mathcal{S}}) \quad , \quad t(\mathfrak{u}) = t(\mathfrak{s})\}.$$

Then the mapping

$$a: \quad [\mathscr{P}(\tilde{\mathscr{S}})] \to [\mathscr{Q}]\setminus\{[\Phi]\}$$
$$[\mathfrak{s}] \mapsto \tilde{a} \circ t(\mathfrak{u}), \qquad \mathfrak{u} \in [\mathfrak{s}]$$

is bijective, and the classes of equivalent preparations are partially ordered by

$$[\mathfrak{s}] < [\mathfrak{u}] \qquad \text{iff} \qquad a([\mathfrak{s}]) \leqslant a([\mathfrak{u}]).$$

The interpretation of this partial ordering is easily given: If $\mathfrak{u}$ suffices to impose a property ($[\alpha]$) on any physical system, so will $\mathfrak{s}$. But $\mathfrak{s}$ may impose more properties.

Definition. Let $\mathfrak{u} \in \mathscr{P}(\tilde{\mathscr{S}})$. We call $\mathfrak{u}$ 'pure' with respect to $\mathscr{Q}$ if

$$[\mathfrak{s}] < [\mathfrak{u}] \qquad \text{implies} \qquad [\mathfrak{s}] = [\mathfrak{u}].$$

Theorem 3. $\mathfrak{u}$ is pure with respect to $\mathscr{Q}$ if and only if $a([\mathfrak{u}])$ is an atom in $([\mathscr{Q}], \leqslant, \varphi)$.

Proof. The condition is necessary: Let $[\Phi] \neq [\alpha] \leqslant a([\mathfrak{u}])$, and assume $[\alpha] \neq a([\mathfrak{u}])$. Then by Postulate 3 we have an s,

$$s \in \mathscr{O}(\varphi[\alpha]) \cap \mathscr{O}(a([\mathfrak{u}])).$$

Hence

$$a([\{s\}]) \leqslant a([\mathfrak{u}]) \wedge \varphi[\alpha] \leqslant a([\mathfrak{u}]).$$

But $\mathfrak{u}$ is pure, so

$$a([\{s\}]) = a([\mathfrak{u}]) \leqslant \varphi[\alpha].$$

Now $a([\mathfrak{u}]) \leqslant \varphi[\alpha] \wedge [\alpha]$ implies $a([\mathfrak{u}]) = [\Phi]$, a contradiction.
Sufficiency is trivial.

By Theorem 3 the lattice $([\mathscr{Q}], \leqslant, \varphi)$ is atomic if we require.

Postulate 4. Let $\mathfrak{s}$ be a set of preparation procedures, then there is a pure set $\mathfrak{u}$, such that $[\mathfrak{u}] < [\mathfrak{s}]$.

The classes of pure sets of preparation procedures are mapped bijectively onto the set of atoms of $([\mathscr{Q}], \leqslant, \varphi)$. If $\mathfrak{u}$ is a pure set and $a([\mathfrak{u}]) = e$ the set of properties present will be $\{([\alpha]) \mid [\alpha] \geqslant e\}$ and the set of properties absent will be $\{(\varphi[\alpha]) \mid [\alpha] \geqslant e\} = \{([\gamma]) \mid [\gamma] \leqslant \varphi e\}$.

Are we right to assume any system prepared by an s, $[\{s\}] = [\mathfrak{u}]$, $\mathfrak{u}$ pure with respect to $\mathscr{Q}$, to be in the same pure state? The first argument against such an assumption is that this concept of pure states depends on $\mathscr{Q}$ because any question not included in $\mathscr{Q}$ may distinguish between systems prepared by different procedures s_i, $[\{s_i\}] = [\mathfrak{u}]$. It is necessary therefore to include into $\mathscr{Q}$ any question possible in nature. This is done by Piron. A second argument against the above assumption is that different s_i may be distinguished by different probabilities for the occurrence of yes if a question β, $a([\mathfrak{u}]) \nleqslant [\beta]$ and $[\beta] \nleqslant \varphi a([\mathfrak{u}])$ is measured.

By abuse of language we shall call any system prepared by an s, $[\{s\}] = [\mathfrak{u}]$, $\mathfrak{u}$ pure, to be in the same pure state with respect to $\mathscr{Q}$. By the arguments just given, we note that this concept of pure state cannot serve as a suitable basis for assuming all systems in the same pure state to possess 'the same properties'.

Let us consider a pure state which corresponds to the atom e in $[\mathscr{Q}]$ and a property $([\alpha])$. Then one of three cases can occur: First, $e \leqslant [\alpha]$, then $([\alpha])$ is present; second, $[\alpha] \leqslant \varphi e$, then $([\alpha])$ is absent; third, $e \nleqslant [\alpha]$ and $[\alpha] \nleqslant \varphi e$, we then call $([\alpha])$ possible.

VI

Postulates 1 through 4 impose the propositional system $([\mathscr{Q}], \leqslant, \varphi)$ to be a complete, orthocomplemented, weakly modular, atomic lattice. A coordinatisation, which brings in the linear structure of usual quantum mechanics, is possible if the covering law holds in $([\mathscr{Q}], \leqslant, \varphi)$.

Piron suggests that structure on $([\mathscr{Q}], \leqslant, \varphi)$ by the requirement that any proposition contains an ideal question of the first kind. Ideal questions and questions of the first kind (Piron, 1972, p. 296) are much more special concepts than those questions (Piron, 1972, p. 290) which we have used until now. In fact, ideal questions and questions of the first kind are to be understood as preparative measurements or operations on states.

In classical physics no problems arise from the assumption of ideal measurements of the first kind, which may be considered as a principle of classical observation. But measurements on quantum systems pre-suppose an interaction process prior to any observation which cannot be described in terms of classical physics.

Generally, the quantum object becomes part of the measuring apparatus wher a proposition is measured. Further experiments on the same physical system are then impossible. Moreover, as in general the system is not isolated after the measurement, it makes no sense to speak of its state after the measurement.

Preparative measurements arise as a special case. Assume preparative measurements to be possible for any proposition. Wigner (1952) has shown that such measurements cannot be ideal and of the first kind for certain propositions if there are universal additive conservation laws. So we will look for another motivation for the covering law.

Typical for quantum systems is the superposition principle, which is well founded by diffraction experiments and basic to most calculations of quantum mechanics which are confirmed by experience. We formulate a superposition postulate which can serve as a realistic basis for imposing the covering law.

Consider a property $([\alpha])$ and a pure set of preparation procedures $\mathfrak{u}$, such that $a([\mathfrak{u}]) \nleqslant [\alpha]$ and $a([\mathfrak{u}]) \nleqslant \varphi[\alpha]$. Then both $([\alpha])$ and $(\varphi[\alpha])$ are possible in this pure state with respect to $\mathscr{Q}$. We call '*superposition postulate*' the assumption that there are two other pure sets of preparation procedures, $\mathfrak{s}_i$ $(i = 1, 2)$, such that $a([\mathfrak{s}_1]) \leqslant [\alpha], a([\mathfrak{s}_2]) \leqslant \varphi[\alpha]$, and

$$a([\mathfrak{u}]) \vee a([\mathfrak{s}_1]) = a([\mathfrak{u}]) \vee a([\mathfrak{s}_2]) = a([\mathfrak{s}_1]) \vee a([\mathfrak{s}_2]) \qquad (+)$$

holds true.

Clearly, the superposition postulate is different from the superposition principle formulated by Jauch (1968, p. 106). This states, for an irreducible part of the propositional system, that, given any two different atoms, let them be given by $a([u])$ and $a([s_1])$, then there is a third, different from both, let it be given by $a([s_2])$, such that (+) holds true. In contrast, our superposition postulate applies only if $a([u]) \nleqslant \varphi a([s_1])$ and then requires the existence of an atom $a([s_2]) \leqslant \varphi a([s_1])$. The superposition postulate can be stated irregardless of whether or not the lattice has trivial centre. If $[\alpha]$ is in the centre and e is an atom, then $e \nleqslant [\alpha]$ and $e \nleqslant \varphi[\alpha]$ is impossible. So the superposition postulate holds trivially if $[\alpha]$ is a central element. On the other hand, if $e \nleqslant [\beta]$ and $e \leqslant \varphi[\beta]$, let i denote the unique atom of the centre with $e \leqslant i$. Then, if e^n $(n = 1, 2)$ denote atoms in $[\mathscr{Q}]$, $e^1 \vee e = e^2 \vee e = e^1 \vee e^2$ implies $e^1 \leqslant i$ and $e^2 \nleqslant i$. In order to prove this, assume $e^1 \leqslant j, j \neq i$, j atom of the centre. Then $(e^1 \vee e) \wedge j = (e^1 \wedge j) \vee (e \wedge j) = e^1$ and this is equal to $(e^2 \vee e) \wedge j = e^2 \wedge j$. But $e^1 = e^2 \wedge j$ implies $e^1 = e^2$, thus $e^1 = e^2 = e$, a contradiction. So the superposition postulate in the non-trivial case reduces to an irreducible component of $([\mathscr{Q}], \leqslant, \varphi)$ in a natural way.†

We have the following theorem, which is proved in the appendix.

Theorem 4. Assume Postulates 1 and 2, and the complete orthocomplemented lattice to be atomic. Then the superposition postulate is equivalent to weak modularity together with the covering law.

So we can impose on $([\mathscr{Q}], \leqslant, \varphi)$ the covering law to hold by the requirement of the superposition postulate. Then also the superposition principle will hold in any irreducible component of $([\mathscr{Q}], \leqslant, \varphi)$.

The superposition postulate holds trivially for classical systems but does not make any statement. So a motivation can hardly be derived from everyday experience. The only thing one can state is that it works in quantum mechanical computations and gives an explanation for diffraction experiments. It comes into play if the distributive law does not hold and one may ask whether it is necessary. We know that ideal questions of the first kind can be described by lattice operations if the superposition postulate holds. A more interesting problem is whether measuring processes can be formulated at all if the superposition postulate does not hold.

In a subsequent paper we shall give a formulation of measuring processes in terms of propositional systems. The covering law will not be required for the propositional calculi of object and apparatus. Observations on the apparatus will be described by questions in the sense used here. We derive a necessary and sufficient criterion for a question ϵ to have property (ii) of Section III, i.e. $[\nu\epsilon] = \varphi[\epsilon]$, and a theorem which connects commensurability and compatibility. So it makes sense to consider physical systems the propositional calculi of which do not fulfil the superposition postulate.

† For details of decompositions of orthocomplemented atomic lattices see MacLaren (1964).

Appendix

We verify the statements (i) through (vi) of Section III by the following steps:

(A1) If $\beta \in \varphi[\alpha]$, then for any $\gamma \in \mathscr{F}([\alpha])$ we have $[\nu\delta] \leqslant [\beta]$.
This is obvious.

(A2) For any $\beta \in \varphi[\alpha]$ we have $[\alpha] \wedge [\beta] = [\Phi]$.

There is a $\delta \in \mathscr{F}([\alpha])$ with $[\nu\delta] = \varphi[\alpha]$. Hence $[\beta] = [\nu\delta]$, and $[\alpha] \leqslant [\delta]$. This proves (A2).

(A3) We have $[\alpha] \leqslant \varphi^2[\alpha]$.

By Postulate 1 there is a $\delta \in \mathscr{F}([\alpha])$ with $\nu\delta \in \varphi[\alpha]$ and a $\epsilon \in \mathscr{F}(\varphi[\alpha])$ with $\nu\epsilon \in \varphi^2[\alpha]$. So $[\nu\gamma] \leqslant [\nu\delta]$ for any $\gamma \in \mathscr{F}([\alpha])$, and $[\nu\kappa] \leqslant [\nu\epsilon]$ for any $\kappa \in \mathscr{F}(\varphi[\alpha])$. Now $\nu\delta \in \mathscr{F}(\varphi[\alpha])$, thus $[\nu\nu\delta] = [\delta] \leqslant [\nu\epsilon]$. From $\delta \in \mathscr{F}([\alpha])$ then follows $[\alpha] \leqslant [\delta] \leqslant [\nu\epsilon] = \varphi^2([\alpha])$. This proves (A3).

(A4) For any $\alpha, \alpha \in \mathscr{Q}$, there is a $\epsilon, \epsilon \in \mathscr{Q}$, such that $\epsilon \in \varphi^2[\alpha]$ and $\nu\epsilon \in \varphi[\alpha]$.

For any $\alpha \in \mathscr{Q}$ there is an $\tilde{\epsilon} := \nu\epsilon \in \mathscr{F}(\varphi[\alpha])$ with $\nu\tilde{\epsilon} = \epsilon \in \varphi^2[\alpha]$, which means $\epsilon \in \mathscr{F}([\alpha])$. Then Postulate 1 requires $[\nu\epsilon] \leqslant \varphi[\alpha]$. Because of $\nu\epsilon \in \mathscr{F}(\varphi[\alpha])$ we get $\nu\epsilon \in \varphi[\alpha]$.

(A5) We have $\varphi^m[\alpha] = \varphi^{m-2}[\alpha]$ if $m \geqslant 3$.
This becomes clear by the definition of $\varphi[\alpha]$ and because of $\varphi^2[\alpha] \geqslant [\alpha]$.

(A6) $[\alpha] \leqslant [\beta]$ implies $\varphi[\beta] \leqslant \varphi[\alpha]$.

We have $\mathscr{F}([\beta]) \subseteq \mathscr{F}([\alpha])$. Hence $\varphi[\beta] \leqslant \varphi[\alpha]$ according to the definition of φ.

(A7) $\varphi[\beta] \leqslant \varphi[\alpha]$ is equivalent to $\varphi^2[\alpha] \leqslant \varphi^2[\beta]$.
This is obvious by (A5) and (A6).

(A8) We have

$$\varphi([\alpha] \vee [\beta]) \leqslant \varphi[\alpha] \wedge \varphi[\beta],$$

and

$$\varphi([\alpha] \wedge [\beta]) \geqslant \varphi[\alpha] \vee \varphi[\beta].$$

These formulas are easily derived from (A6).

(A9) We have

$$\varphi^2([\alpha] \wedge [\beta]) \leqslant \varphi(\varphi[\alpha] \vee \varphi[\beta]) \leqslant \varphi^2[\alpha] \wedge \varphi^2[\beta],$$

and

$$\varphi^2([\alpha] \vee [\beta]) \geqslant \varphi(\varphi[\alpha] \wedge \varphi[\beta]) \geqslant \varphi^2[\alpha] \vee \varphi^2[\beta].$$

Both formulas are easily derived by repeated use of (A8).

(A10) $\varphi^2([\alpha] \vee [\beta]) = \varphi(\varphi[\alpha] \wedge \varphi[\beta]) = \varphi^2(\varphi^2[\alpha] \vee \varphi^2[\beta])$.

Using (A3) we get $\varphi^2([\alpha] \vee [\beta]) \leqslant \varphi^2(\varphi^2[\alpha] \vee \varphi^2[\beta])$. Moreover, applying (A5) and (A6) to (A9) we get $\varphi(\varphi[\alpha] \wedge \varphi[\beta]) \geqslant \varphi^2(\varphi^2[\alpha] \vee \varphi^2[\beta])$. These formulas, together with (A9), provide the assertion.

(A11) $\varphi[\alpha] \wedge \varphi[\beta] \leqslant \varphi[\gamma]$ implies $\varphi^2(\varphi^2[\alpha] \vee \varphi^2[\beta]) \geqslant \varphi^2[\gamma]$.
This is easily proved if one applies (A6) and then (A10).

(A12) We have $\varphi[\Phi] = [I]$ and $\varphi[I] = [\Phi]$.

By $\Phi \in \mathscr{F}([\Phi])$ we have $\varphi[\Phi] \geqslant [\nu\Phi] = [I]$. By $\mathscr{F}([I]) = \{I\}$ we have $\varphi[I] = [\nu I] = [\Phi]$.

(A13) We have $\varphi[\alpha] \wedge \varphi^2[\alpha] = [\Phi]$ and $\varphi^2(\varphi[\alpha] \vee \varphi^2[\alpha]) = [I]$.

The first formula is clear by (A2). Application of (A5), (A11), and (A12) furnishes $\varphi[\Phi] = [I] \leqslant \varphi^2(\varphi^3[\alpha] \vee \varphi^4[\alpha]) = \varphi^2(\varphi[\alpha] \vee \varphi^2[\alpha])$.

By (A3), (A4), (A5), (A7), and (A13) the statements of Section III are proved.

We now approve Theorem 4 of Section VI. Firstly we show, that the superposition postulate is sufficient to impose weak modularity and the covering law. We assume for any question α and any atom e, $e \nleqslant \varphi[\alpha]$, $e \nleqslant [\alpha]$, that there are two atoms $e^1, e^1 \leqslant [\alpha]$, and $e^2, e^2 \leqslant \varphi[\alpha]$, such that

$$e \vee e^1 = e \vee e^2 = e^1 \vee e^2$$

holds true. We use these notations in the following discussion of the superposition postulate.

(A14) Let $[\alpha] \neq [\Phi]$, and $[\beta] \neq [\Phi]$. Then $[\beta] \nleqslant \varphi[\alpha]$ implies $([\beta] \vee \varphi[\alpha]) \wedge [\alpha] \neq [\Phi]$.

To prove this, recall that $([\beta] \vee \varphi[\alpha]) \wedge [\alpha] \geqslant ([\beta] \wedge [\alpha]) \vee ([\alpha] \wedge \varphi[\alpha]) = [\beta] \wedge [\alpha]$ holds in any orthocomplemented lattice true. In the non-trivial case, $[\alpha] \wedge [\beta] = [\Phi]$, we have by the assumption $[\beta] \nleqslant \varphi[\alpha]$ the existence of at least one atom e, $e \leqslant [\beta]$, $e \nleqslant [\alpha]$, and $e \nleqslant \varphi[\alpha]$. So the superposition postulate applies, leading to $([\beta] \vee \varphi[\alpha]) \wedge [\alpha] \geqslant (e \vee e^2) \wedge e^1 = (e \vee e^1) \wedge e^1 = e^1 \neq [\Phi]$.

(A15) Let $[\alpha] \neq [\Phi]$, $[\alpha] \leqslant [\beta]$, $[\alpha] \neq [\beta]$. Then $\varphi[\alpha] \wedge [\beta] \neq [\Phi]$.

We have $[\beta] \neq [\Phi]$ and $\varphi[\alpha] \neq \varphi[\beta] \geqslant [\Phi]$. Moreover, $[\beta] \nleqslant [\alpha] = \varphi^2[\alpha]$. So by (A14) $([\beta] \vee \varphi^2[\alpha]) \wedge \varphi[\alpha] = [\beta] \wedge \varphi[\alpha] \neq [\Phi]$.

(A16) Let $[\alpha] \leqslant [\beta]$; then $\varphi[\alpha] \wedge [\beta] = [\Phi]$ is equivalent to $[\alpha] = [\beta]$.

The case $[\alpha] = [\Phi]$ is trivial. In case of $[\alpha] \neq [\Phi]$ the statement is easily proved with the help of (A15).

(A17) $[\alpha] \leqslant [\beta]$ implies $[\alpha] \vee ([\beta] \wedge \varphi[\alpha]) = [\beta]$, and $[\beta] \wedge ([\alpha] \vee \varphi[\beta]) = [\alpha]$.

(A16) is just the criterion of weak modularity stated by Theorem 2 of Section IV. This proves (A17).

(A18) Let us be given an atom e and a question $[\alpha]$, $[\alpha] \neq [\Phi]$. Then $e \nleqslant \varphi[\alpha]$ implies $(e \vee \varphi[\alpha]) \wedge [\alpha]$ to be an atom.

In case $e \leqslant [\alpha]$ the statement holds trivially by (A17). In the non-trivial case, $e \nleqslant [\alpha]$, the superposition postulate applies to provide $(e \vee \varphi[\alpha]) \wedge [\alpha] \geqslant e^1$ (recall the end of the proof of (A14)). By $e \leqslant e^1 \vee e^2$ and $e^2 \leqslant \varphi[\alpha]$ we have on the other hand $(e \vee \varphi[\alpha]) \wedge [\alpha] \leqslant (e^1 \vee e^2 \vee \varphi[\alpha]) \wedge [\alpha] = (e^1 \vee \varphi[\alpha]) \wedge [\alpha] = e^1$ because of $e^1 \leqslant [\alpha]$ and (A17).

(A19) Let us be given an atom e, $e \nleqslant [\alpha]$, then $[\alpha] \leqslant [\beta] \leqslant [\alpha] \vee e$ implies that either $[\beta] = [\alpha]$ or $[\beta] = [\alpha] \vee e$.

We have $[\beta] \wedge \varphi[\alpha] \leqslant (e \vee [\alpha]) \wedge \varphi[\alpha]$ which is an atom by (A18). So either $[\beta] \wedge \varphi[\alpha] = [\Phi]$ or $[\beta] \wedge \varphi[\alpha] = (e \vee [\alpha]) \wedge \varphi[\alpha]$. In the first case, $[\beta] = [\alpha]$ is implied by (A16). In the second case, (A17) provides $[\beta] = ([\beta] \wedge \varphi[\alpha]) \vee [\alpha] = ((e \vee [\alpha]) \wedge \varphi[\alpha]) \vee [\alpha] = (e \vee [\alpha])$.

Statements (A17) and (A19) provide the first part of the proof of Theorem 4.

We assume now, conversely, weak modularity and the covering law, i.e. we use (A16), (A17), and (A19) to derive the superposition postulate. For this purpose, we make technical use of compatibility theory in the weak modular case. We write $[\gamma] \leftrightarrow [\delta]$ iff $[\gamma]$ and $[\delta]$ are compatible, i.e. belong to a boolean sublattice of $([\mathcal{Q}], \leqslant, \varphi)$. We need the following results (Piron, 1964, Appendix, Theorems VII and VIII):

(A20) (1) $[\gamma] \leqslant [\delta]$ implies $[\gamma] \leftrightarrow [\delta]$.
(2) $[\gamma] \leftrightarrow [\delta]$ implies $[\gamma] \leftrightarrow \varphi[\delta]$.
(3) If $[\gamma] \leftrightarrow [\delta_n]$ $(n = 1, 2)$, then
$[\gamma] \vee ([\delta_1] \wedge [\delta_2]) = ([\gamma] \vee [\delta_1]) \wedge ([\gamma] \vee [\delta_2])$,
$[\gamma] \wedge ([\delta_1] \vee [\delta_2]) = ([\gamma] \wedge [\delta_1]) \vee ([\gamma] \wedge [\delta_2])$.

Now let α be any question and e be any atom with $e \nleqslant [\alpha]$, and $e \nleqslant \varphi[\alpha]$. Then

(A21) $e^1 := (e \vee \varphi[\alpha]) \wedge [\alpha]$, and $e^2 := (e \vee [\alpha]) \wedge \varphi[\alpha]$ are atoms.

We prove this for e^2: Assume $e^2 = [\Phi]$, then (A16) applies to give $e \vee [\alpha] = [\alpha]$, a contradiction. Hence $e^2 \neq [\Phi]$.

Let us be given an atom d, $d \leqslant e^2$, then $[\alpha] \leqslant d \vee [\alpha] \leqslant e^2 \vee [\alpha]$. Use of (A20) provides $e^2 \vee [\alpha] = e \vee [\alpha]$, so $[\alpha] \leqslant d \vee [\alpha] \leqslant e^2 \vee [\alpha] = e \vee [\alpha]$. From $d \leqslant e^2 \leqslant \varphi[\alpha] \neq [I]$ we have $[\alpha] \neq d \vee [\alpha]$. So (A19) gives $e \vee [\alpha] = d \vee [\alpha] = e^2 \vee [\alpha]$. From this and (A17) $d = e^2$ is easily concluded, which proves e^2 to be an atom. Similarly one proves the statement for e^1. As a by-product we have:

(A22) $e \vee [\alpha] = e^2 \vee [\alpha]$, and $e \vee \varphi[\alpha] = e^1 \vee \varphi[\alpha]$.

Now the formula

(A23) $$e \vee e^1 = e \vee e^2 = e^1 \vee e^2$$

is easily derived by the use of (A20) and (A22): From $e^1 \leqslant [\alpha]$ we have

$$e^1 \vee e^2 = ((e \vee [\alpha]) \wedge e^1) \vee ((e \vee [\alpha]) \wedge \varphi[\alpha]).$$

(A20) and (A22) lead to $(e \vee [\alpha]) \leftrightarrow e^1$, $(e \vee [\alpha]) \leftrightarrow \varphi[\alpha]$, and

$$e^1 \vee e^2 = (e \vee [\alpha]) \wedge (e^1 \wedge \varphi[\alpha]).$$

We now have, from (A22) and (A20), that $(e^1 \vee \varphi[\alpha]) \leftrightarrow e$, and $(e^1 \vee \varphi[\alpha]) \leftrightarrow [\alpha]$, and

$$e^1 \vee e^2 = (e \wedge (e^1 \vee \varphi[\alpha])) \vee ([\alpha] \wedge (e^1 \vee \varphi[\alpha])).$$

Use of (A22) provides

$$e^1 \vee e^2 = e \vee e^1.$$

Analogously one checks $e^1 \vee e^2 = e \vee e^2$.

Definition (A21) and Formula (A23) show the superposition postulate to hold true, which completes the proof of Theorem 4.

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